

# FINITE DIMENSIONAL FOKKER-PLANCK EQUATIONS FOR CONTINUOUS TIME RANDOM WALK LIMITS

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**ABSTRACT.** Continuous Time Random Walk(CTRW) is a model where particle's jumps in space are coupled with waiting times before each jump. A Continuous Time Random Walk Limit(CTRWL) is obtained by a limit procedure on a CTRW and can be used to model anomalous diffusion. The distribution  $p(dx, t)$  of a CTRWL  $X_t$  satisfies a Fractional Fokker-Planck Equation(FFPE). Since CTRWLs are usually not Markovian, their one dimensional FFPE is not enough to completely define them. In this paper we find the FFPEs of the distribution of  $X_t$  at multiple times, i.e. the distribution of the random vector  $(X_{t_1}, \dots, X_{t_n})$  for  $t_1 < \dots < t_n$  for a large class of CTRWLs. This allows us to define CTRWLs by their finite dimensional FFPEs.

## 1. INTRODUCTION

CTRW models the movement of a particle in space, where the  $k$ 'th jump  $J_k$  of the particle in space succeeds the  $k$ 'th waiting time  $W_k$ . We let  $N_t = \sup\{k : T_k \leq t\}$  where  $T_k = \sum_{i=1}^k W_i$ , if  $T_1 > t$  then we set  $N_t$  to be 0.  $N_t$  is just the number of jumps of the particle up to time  $t$ . Then

$$X'_t = \sum_{k=1}^{N_t} J_k,$$

is the CTRW associated with the time-space jumps  $\{(J_k, W_k)\}_{k \in \mathbb{N}}$ . Let us now assume that  $\{J_k\}$  and  $\{W_k\}$  are independent i.i.d sequences of random variables. In order to model the long time behavior of the CTRW we write  $\{(J_k^c, W_k^c)\}_{k \in \mathbb{N}}$  for  $c > 0$ . Here the purpose of  $c$  is to facilitate the convergence of the trajectories of  $\{(J_k^c, W_k^c)\}_{k \in \mathbb{N}}$  weakly on a proper space. More precisely, we let  $\mathcal{D}([0, \infty), \mathbb{R}^2)$  be the space of càdlàg functions  $f : [0, \infty) \rightarrow \mathbb{R}^2$  equipped with the Skorokhod  $J_1$  topology. We assume that

$$(S_u^c, T_u^c) = \sum_{k=1}^{\lfloor cu \rfloor} (J_k^c, W_k^c) \Rightarrow (A_u, D_u) \quad c \rightarrow \infty,$$

where  $\Rightarrow$  denotes weak convergence of measures with respect to the  $J_1$  topology. We further assume that the processes  $A_t$  and  $D_t$  are independent Lévy processes and that  $D_t$  is a strictly increasing subordinator. Denote by  $X_t^c$  the CTRW associated with  $\{(J_k^c, W_k^c)\}_{k \in \mathbb{N}}$ . We then have ([14, Theorem 3.6] and [13, Lemma 2.4.5])

$$(1.1) \quad X_t^c \Rightarrow X_t = A_{E_t} \quad c \rightarrow \infty,$$

where  $E_t = \inf\{s : D_s > t\}$  is the inverse of  $D_t$  and  $\Rightarrow$  means weak convergence on  $\mathcal{D}([0, \infty), \mathbb{R})$  equipped with the  $J_1$  topology. It is well known that  $X_t$  is usually not Markovian, a fact that makes the task of finding basic properties of  $X_t$  nontrivial. One such task is finding the finite dimensional distributions(FDDs) of the process  $X_t$ , i.e.  $P(X_{t_1} \in dx_1, \dots, X_{t_n} \in dx_n)$ . In [10], Meerschaert and Straka used a semi-Markov approach to find the FDDs for a large class of CTRWL. It turns out

that the discrete regeneration times of  $X_t^c$  converge to a set of points where  $X_t$  is renewed. Once we know the next time of regeneration of  $X_t$ , we no longer need older observations in order to determine the future behavior of  $X_t$ . More mathematically, denote by  $R_t = D_{E_t} - t$  the time left before regeneration of  $X_t$  then  $(X_t, R_t)$  is a Markov process. One can then use the transition probabilities of  $(X_t, R_t)$  along with the Chapman-Kolmogorov Equations in order to find  $P(X_{t_1} \in dx_1, \dots, X_{t_n} \in dx_n)$  for  $t_1 < \dots < t_n$  and  $n \in \mathbb{N}$ . This method was used in [5] in order to find the FDD of the aged process  $X_t^{t_0} = X_t - X_{t_0}$ . It is well known ([8, Section 4.5]) that the one dimensional distribution  $p(dx, t) = P(X_t \in dx)$  satisfies a FFPE. Once again, as  $X_t$  is non Markovian the FFPE satisfied by  $p(dx, t)$  is not enough to fully describe  $X_t$  (as it does when  $X_t$  is Markovian). Hence, a dual problem to finding the FDDs is that of finding the finite dimensional FFPEs of the FDDs of  $X_t$ . In this paper we obtain the finite dimensional FFPEs for a large class of CTRWL. The results generalize the well known one dimensional FFPE of CTRW([7]) as well as results in the finite dimensional case([2],[3]).

In Section 2 we present relevant mathematical background for this paper and prepare the way for our main result. It is divided into 4 subsections; Subsection 2.1 introduces the notation to be used throughout the paper, Subsection 2.2 presents the Caputo and Riemann-Liouville fractional derivatives, Subsection 2.3 establishes results regarding pseudo-differential operators(PDOs) on certain multivariable functions which facilitate the proof of Theorem 1 and Subsection 2.4 presents briefly the work in [10] upon which we establish our results.

Section 3 presents our main results; Theorem 1 gives the finite dimensional FFPEs of  $E_t$ , Corollary 1 states the finite dimensional FFPEs of the process  $X_t = A_{E_t}$  where the outer process  $A_t$  and the subordinator  $D_t$  are independent. Finally, Corollary 2 gives the finite dimensional FFPEs of the coupled case. Section 4 compares our results with the well known finite dimensional case.

In Section 5 we show that if  $\xi(-k, s)$  is the symbol of a PDO on a suitable Banach space then  $\xi(-\sum_{i=1}^n k_i, \sum_{i=1}^n s_i)$  is also a symbol of a PDO on another Banach space. This complements the results in Section 3.

## 2. MATHEMATICAL BACKGROUND

**2.1. Notations.** A well known method of solving partial differential equations of distributions  $p(dx_1, \dots, dx_n; t_1, \dots, t_n)$  on  $\mathbb{R}^n$  is taking the Fourier Transform(FT) of the distribution with respect to the spatial variables and then the Laplace Transform(LT) with respect to the time variables. This is referred to as the Fourier Laplace Transform(FLT) of  $p(dx_1, \dots, dx_n; t_1, \dots, t_n)$ . More generally, for  $m, n \in \mathbb{N}$  let  $f(dx_1, \dots, dx_m; t_1, \dots, t_n)$  be a finite measure on  $\mathbb{R}^m$  for every  $\mathbf{t} = (t_1, \dots, t_n)$  s.t

$0 < t_1 \leq \dots \leq t_n$  and assume that  $\int_{\mathbf{x} \in A} f(dx_1, \dots, dx_m; t_1, \dots, t_n)$  is measurable as a function of  $\mathbf{t}$  for every measurable  $A \subset \mathbb{R}^m$ . We denote the FT of  $f$  by

$$\tilde{f}(k_1, \dots, k_m; t_1, \dots, t_n) = \int_{x_1 \in \mathbb{R}} \dots \int_{x_m \in \mathbb{R}} e^{-i \sum_{j=1}^m k_j x_j} f(dx_1, \dots, dx_m; t_1, \dots, t_n).$$

When  $f$  has density  $f(x_1, \dots, x_m; t_1, \dots, t_n)$  we denote the LT of  $f$  by

$$\hat{f}(x_1, \dots, x_m; s_1, \dots, s_n) = \int_{t_1=0}^{\infty} \dots \int_{t_n=0}^{\infty} e^{-\sum_{j=1}^n s_j t_j} f(x_1, \dots, x_m; t_1, \dots, t_n) dt_1 \dots dt_n.$$

The FLT of  $f$  is

$$\bar{f}(k_1, \dots, k_m; s_1, \dots, s_n) = \int_{t_1=0}^{\infty} \cdots \int_{t_n=0}^{\infty} \int_{x_1 \in \mathbb{R}} \cdots \int_{x_n \in \mathbb{R}} e^{-i \sum_{j=1}^m k_j x_j - \sum_{j=1}^n s_j t_j} f(dx_1, \dots, dx_m; t_1, \dots, t_n) dt_1 \cdots dt_n.$$

We also denote by  $\tilde{f}$  the FT of  $f$  with respect to some of its spatial variables, therefore,  $\tilde{f}(dx_1, k_2; t_1, t_2)$  is the FT of  $f$  w.r.t  $x_2$ . Similarly,  $\hat{f}(dx_1, dx_2; s_1, t_2)$  is the LT of  $f$  w.r.t  $t_1$  and  $\bar{f}(k_1, dx_2; s_1, t_2)$  is the FLT of  $f$  w.r.t  $x_1$  and  $t_1$ . When using the hat symbol is cumbersome we also use  $\hat{f} = \mathcal{L}(f)$ . We occasionally use bold font to represent the vector  $\mathbf{x} = (x_1, \dots, x_n)$  where the size of the vector is clear.

**2.2. Caputo and Riemann-Liouville Fractional Derivatives.** The Riemann-Liouville(RL) fractional derivative of index  $0 < \alpha < 1$  is given by

$$(2.1) \quad \mathfrak{D}_t^\alpha f(t) = \frac{\partial}{\partial t} \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-r)^{-\alpha} f(r) dr,$$

for a suitable function  $f$  defined on  $\mathbb{R}_+$ . When the variable with respect to which we take the derivative is obvious we drop the subscript and just write  $\mathfrak{D}^\alpha f(t)$ . It can be verified that the LT of (2.1) is

$$\widehat{\mathfrak{D}^\alpha f}(s) = s^\alpha \hat{f}(s).$$

Hence, the RL derivative is a PDO of symbol  $s^\alpha$ . Caputo's derivative is obtained by moving the derivative in (2.1) under the integral to obtain

$$(2.2) \quad \mathbb{D}_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-r)^{-\alpha} \frac{\partial}{\partial r} f(r) dr.$$

The LT of (2.2) is

$$\widehat{\mathbb{D}^\alpha f}(s) = s^\alpha \hat{f}(s) - s^{\alpha-1} f(0^+).$$

We denote the classic derivative by  $\frac{\partial}{\partial t} = \mathbb{D}^1$ , and note that  $\mathbb{D}^1 = \mathfrak{D}^1$  iff  $f(0^+) = 0$ . For simplicity we drop the superscript and write  $\frac{\partial}{\partial t} = \mathbb{D}$  (or  $\frac{\partial}{\partial t} = \mathfrak{D}$  when that is the case).

**2.3. Pseudo-differential operators of multivariable functions.** Here we investigate the PDOs acting on measures  $f(dx_1, \dots, dx_n)$  on  $\mathbb{R}_+^n$  with support in  $A^n = \{\mathbf{x} : 0 \leq x_1 \leq x_2 \leq \dots \leq x_n\}$  with LT  $\hat{f}$ . Let  $\mathbf{k} = (k_1, \dots, k_l)$  be a strictly increasing  $l$ -tuple where  $1 \leq k_i \leq n$  for  $1 \leq i \leq l \leq n$  and s.t  $k_1 = 1$ . We shall sometimes abuse notation and write  $i \in \mathbf{k}$  where we mean that  $i = k_j$  for some  $1 \leq j \leq l$ . We also write  $\mathbf{k}^c$  for the increasing vector s.t  $i \in \mathbf{k}^c$  iff  $2 \leq i \leq n$  and  $i \notin \mathbf{k}$ . If  $\mathbf{x}$  is a vector of length  $n$  we write  $\mathbf{x}_{\mathbf{k}}$  for the vector of length  $l$  whose  $i$ 'th element is  $x_{k_i}$ . Let  $A_{\mathbf{k}}^n$  be the set of all  $\mathbf{x} \in A^n$  s.t  $x_{i-1} < x_i$  iff  $i \in \mathbf{k}$  and where  $x_0 = 0$ . For example, for  $n = 3$   $A_{(1,2)}^3 = \{\mathbf{x} : 0 < x_1 < x_2 = x_3\}$ . Since our interest in these distributions comes from the FDDs of the process  $E_t$ , i.e.  $h(dx_1, \dots, dx_n; t_1, \dots, t_n) = P(E_{t_1} \in dx_1, \dots, E_{t_n} \in dx_n)$  we also assume in this subsection that  $f(dx_1, \dots, dx_n)$  can be written as  $f(dx_1, \dots, dx_n) = f(x_{k_1}, \dots, x_{k_l}) \delta_{k_1^c-1}^c(dx_{k_1^c}) \times \dots \times \delta_{k_{n-l}^c-1}^c(dx_{k_{n-l}^c}) dx_{k_1} \cdots dx_{k_l}$  where  $f(x_{k_1}, \dots, x_{k_l})$  is absolutely continuous(a.c) in each of its variables, i.e.  $x_{k_i} \rightarrow f(x_1, \dots, x_n)$  is a.c with respect to Lebesgue measure on  $\mathbb{R}$  for each  $1 \leq i \leq l$ .

We occasionally refer to such  $f$  as a.c, not to be confused with the concept of a.c measure on  $\mathbb{R}^n$ . We abbreviate by writing

$$(2.3) \quad f_{\mathbf{k}}(d\mathbf{x}) = f(\mathbf{x}_{\mathbf{k}}) \delta_{\mathbf{k}^c-1}(d\mathbf{x}_{\mathbf{k}^c}) d\mathbf{x}_{\mathbf{k}},$$

so that  $\mathbf{k}$  points out the indices for which  $f$  has absolutely continuous density. For example,  $f_{(1,2,4)}(dx_1, dx_2, dx_3, dx_4)$  can be written as  $f(x_1, x_2, x_4) \delta_{x_2}(dx_3) dx_1 dx_2 dx_4$ . To motivate this assumption cf. (3.1) and note that by (2.8)  $h(dx_1, \dots, dx_n; t_1, \dots, t_n)$  is of the form  $f(\mathbf{x}_{\mathbf{k}})$  on  $A_{\mathbf{k}}^n$ . The set  $A_{\mathbf{k}}^n$  is a manifold of dimension  $l$ , and represents the event where the process  $E_t$  has been stuck at the point  $x_i$  since the time  $t_{i-1}$  to  $t_i$  for  $i \notin \mathbf{k}$ . For example,  $A_{(1,3)}^4$  represents the event  $\{E_{t_1} = x_1 \in (0, \infty), E_{t_2} = x_1, E_{t_3} = x_3 \in (x_1, \infty), E_{t_4} = x_3\}$ , and it helps to think of  $\mathbf{k}$  as the indices of mobilized points of the particle. Let us define a derivative operator on  $f_{\mathbf{k}}$  distributions. We define the derivative operator to be

$$\mathbb{D}_{\mathbf{x}} f_{\mathbf{k}}(d\mathbf{x}) = \sum_{i=1}^l \frac{\partial}{\partial x_{k_i}} f(\mathbf{x}_{\mathbf{k}}) \delta_{\mathbf{k}^c-1}(d\mathbf{x}_{\mathbf{k}^c}) d\mathbf{x}_{\mathbf{k}}.$$

For example, if  $f_{(1,2)}(d\mathbf{x}) = f(x_1, x_2) \delta_{x_2}(dx_3) dx_1 dx_2$  then

$$\begin{aligned} \mathbb{D}_{\mathbf{x}} f_{(1,2)}(d\mathbf{x}) &= \frac{\partial}{\partial x_1} f_{(1,2)}(x_1, x_2) \delta_{x_2}(dx_3) dx_1 dx_2 \\ &\quad + \frac{\partial}{\partial x_2} f_{(1,2)}(x_1, x_2) \delta_{x_2}(dx_3) dx_1 dx_2. \end{aligned}$$

Note that  $\mathbb{D}_{\mathbf{x}}$  is well defined as we assume that  $f_{\mathbf{k}}$  has a.c density in  $x_i$  for  $i \in \mathbf{k}$ . We also assume that  $\lim_{x_{k_l} \rightarrow \infty} e^{-x_{k_l}} f(x_{k_1}, \dots, x_{k_l}) = 0$  where  $f$  is as in (2.3). This is not a strong assumption as  $f$  has LT.

**Lemma 1.** *Let  $f_{\mathbf{k}}$  be such that  $l = n$ . Then the LT of  $\mathbb{D}_{\mathbf{x}} f(\mathbf{x})$  is*

$$(2.4) \quad \widehat{\mathbb{D}_{\mathbf{x}} f_{\mathbf{k}}}(\mathbf{s}) = \left( \sum_{i=1}^n s_i \right) \widehat{f}_{\mathbf{k}}(s_1, \dots, s_n) - \lim_{x_1 \rightarrow 0^+} \widehat{f}_{\mathbf{k}}(x_1, s_2, \dots, s_n).$$

*Proof.* In the following, we use  $\check{a}_i$  to indicate that  $a_i$  is absent from where it normally should be. Since here  $f_{\mathbf{k}}(d\mathbf{x}) = f(\mathbf{x}) d\mathbf{x}$ , for  $1 \leq i \leq n$  we have

(2.5)

$$\begin{aligned}
& \int_{x_1=0}^{\infty} \int_{x_2=0}^{\infty} \dots \int_{x_n=0}^{\infty} e^{-\langle \mathbf{s}, \mathbf{x} \rangle} \frac{\partial f(\mathbf{x})}{\partial x_i} d\mathbf{x} \\
&= \int_{x_1=0}^{\infty} \dots \int_{x_i=0}^{\check{\infty}} \dots \int_{x_n=0}^{\infty} e^{-s_1 x_1 \dots - s_i \check{x}_i \dots - s_n x_n} \left[ \int_{x_i=0}^{\infty} e^{-s_i x_i} \frac{\partial f(\mathbf{x})}{\partial x_i} dx_i \right] dx_1 \dots d\check{x}_i \dots dx_n \\
&= \int_{x_1=0}^{\infty} \dots \int_{x_i=0}^{\check{\infty}} \dots \int_{x_n=0}^{\infty} e^{-s_1 x_1 \dots - s_i \check{x}_i \dots - s_n x_n} \left[ e^{-s_i x_i} f(\mathbf{x}) \Big|_{(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)}^{(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)} \right. \\
&\quad \left. + s_i \int_{x_i=0}^{\infty} e^{-s_i x_i} f(\mathbf{x}) dx_i \right] dx_1 \dots d\check{x}_i \dots dx_n \\
&= \int_{x_1=0}^{\infty} \dots \int_{x_i=0}^{\check{\infty}} \dots \int_{x_n=0}^{\infty} e^{-s_1 x_1 \dots - s_i \check{x}_i \dots - s_n x_n} \left[ e^{-s_i x_{i+1}} f \left( x_1 \dots, x_{i-1}, \underbrace{x_{i+1}}_{i\text{'th coordinate}}, x_{i+1} \dots, x_n \right) \right. \\
&\quad \left. - e^{-s_i x_{i-1}} f \left( x_1 \dots, x_{i-1}, \underbrace{x_{i-1}}_{i\text{'th coordinate}}, x_{i+1} \dots, x_n \right) + s_i \int_{x_i=0}^{\infty} e^{-s_i x_i} f(x_1, x_2, \dots, x_n) dx_i \right] dx_1 \dots d\check{x}_i \dots dx_n \\
&= \int_{x_{i+1}=0}^{\infty} e^{-(s_i + s_{i+1})x_{i+1}} \hat{f} \left( s_1 \dots, s_{i-1}, \underbrace{x_{i+1}}_{i\text{'th coordinate}}, x_{i+1}, s_{i+2} \dots, s_n \right) dx_{i+1} \\
&\quad - \int_{x_{i-1}=0}^{\infty} e^{-(s_i + s_{i-1})x_{i-1}} \hat{f} \left( s_1 \dots, x_{i-1}, \underbrace{x_{i-1}}_{i\text{'th coordinate}}, s_{i+1}, s_{i+2} \dots, s_n \right) dx_{i-1} + s_i \hat{f}(s_1, s_2, \dots, s_n)
\end{aligned}$$

Note that since  $\lim_{x_n \rightarrow \infty} e^{-x_n} f(x_1, \dots, x_n) = 0$ , summing over the variable  $i$  the first two terms in the last equality in (2.5) cancel out for every  $i \neq 1$ . For  $i = 1$  only the second term in the brackets cancels out and the result follows.  $\square$

**Lemma 2.** *The LT of  $\mathbb{D}_{\mathbf{x}} f_{\mathbf{k}}(\mathbf{x})$  is*

$$\widehat{\mathbb{D}_{\mathbf{x}} f_{\mathbf{k}}}(\mathbf{s}) = \left( \sum_{i=1}^n s_i \right) \hat{f}_{\mathbf{k}}(\mathbf{s}) - \lim_{x_1 \rightarrow 0^+} \hat{f}_{\mathbf{k}}(x_1, s_2, \dots, s_n)$$

*Proof.* Taking the LT of  $f_{\mathbf{k}}(dx_1, \dots, dx_n)$  first w.r.t the indices that are not in  $\mathbf{k}$  we see that

$$(2.6) \quad \widehat{\mathbb{D}_{\mathbf{x}} f_{\mathbf{k}}}(\mathbf{x}_{\mathbf{k}}, \mathbf{s}_{\mathbf{k}^c}) = \int_{\mathbb{R}_+^{n-l}} e^{-\sum_{i \in \mathbf{k}^c} s_i x_i} \mathbb{D}_{\mathbf{x}} f_{\mathbf{k}}(\mathbf{x}),$$

and can be written as

$$\widehat{\mathbb{D}_{\mathbf{x}} f_{\mathbf{k}}}(\mathbf{x}_{\mathbf{k}}, \mathbf{s}_{\mathbf{k}^c}) = \mathbb{D}_{\mathbf{x}} f(\mathbf{x}_{\mathbf{k}}) e^{-\sum_{i=1}^{l-1} \left( \sum_{j=k_i+1}^{k_{i+1}-1} s_j \right) x_{k_i} - \sum_{j=k_l+1}^n s_j x_{k_l}},$$

where  $f$  is an a.c function. It follows that

$$\begin{aligned} \widehat{\mathbb{D}_{\mathbf{x}} f_{\mathbf{k}}}(\mathbf{s}) &= \int_{\mathbb{R}_+^l} e^{-\sum_{i \in \mathbf{k}} s_i x_i} \widehat{\mathbb{D}_{\mathbf{x}} f_{\mathbf{k}}}(\mathbf{x}_{\mathbf{k}}, \mathbf{s}_{\mathbf{k}^c}) d\mathbf{x}_{\mathbf{k}} \\ &= \int_{\mathbb{R}_+^l} e^{-\sum_{i \in \mathbf{k}} s_i x_i} \mathbb{D}_{\mathbf{x}} f(\mathbf{x}_{\mathbf{k}}) e^{-\sum_{i=1}^{l-1} \left( \sum_{j=k_i+1}^{k_{i+1}-1} s_j \right) x_{k_i} - \sum_{j=k_l+1}^n s_j x_{k_l}} d\mathbf{x}_{\mathbf{k}} \\ &= \int_{\mathbb{R}_+^l} e^{-\sum_{i=1}^{l-1} \left( \sum_{j=k_i}^{k_{i+1}-1} s_j \right) x_{k_i} - \sum_{j=k_l}^n s_j x_{k_l}} \mathbb{D}_{\mathbf{x}} f(\mathbf{x}_{\mathbf{k}}) d\mathbf{x}_{\mathbf{k}}. \end{aligned}$$

By Lemma 1 for  $n = l$  we see that

$$\begin{aligned} \widehat{\mathbb{D}_{\mathbf{x}} f_{\mathbf{k}}}(\mathbf{s}) &= \left( \sum_{i=1}^n s_i \right) \int_{\mathbb{R}_+^l} e^{-\sum_{i=1}^{l-1} \left( \sum_{j=k_i}^{k_{i+1}-1} s_j \right) x_{k_i} - \sum_{j=k_l}^n s_j x_{k_l}} f(\mathbf{x}_{\mathbf{k}}) d\mathbf{x}_{\mathbf{k}} \\ &\quad - \lim_{x_1 \rightarrow 0^+} \int_{\mathbb{R}_+^{l-1}} e^{-\sum_{i=2}^{l-1} \left( \sum_{j=k_i}^{k_{i+1}-1} s_j \right) x_{k_i} - \sum_{j=k_l}^n s_j x_{k_l}} f(\mathbf{x}_{\mathbf{k}}) d\mathbf{x}_{\mathbf{k}'}, \end{aligned}$$

where  $\mathbf{k}'$  is just the vector of length  $l-1$  s.t  $k'_i = k_{i+1}$  for  $1 \leq i \leq l-1$ . Since

$$\begin{aligned} \widehat{f_{\mathbf{k}}}(\mathbf{s}) &= \int_{\mathbb{R}_+^n} e^{-\sum_{i=1}^n s_i x_i} f(\mathbf{x}_{\mathbf{k}}) \delta_{\mathbf{k}^c-1}(d\mathbf{x}_{\mathbf{k}^c}) d\mathbf{x}_{\mathbf{k}} \\ &= \int_{\mathbb{R}_+^l} e^{-\sum_{i=1}^{l-1} \left( \sum_{j=k_i}^{k_{i+1}-1} s_i \right) x_{k_i} - \sum_{j=k_l+1}^n s_j x_{k_l}} f(\mathbf{x}_{\mathbf{k}}) d\mathbf{x}_{\mathbf{k}}, \end{aligned}$$

the result follows.  $\square$

If  $f(\mathbf{x})$  is a differentiable function then  $\mathbb{D}_{\mathbf{x}}$  is just the directional derivative along the vector  $v = (1, \dots, 1)$  of size  $n$ . Let  $\Psi_x$  be a PDO on  $\mathbb{R}$  with symbol  $\psi(k)$ . Then  $\psi(\sum_{i=1}^n k_i)$  is a symbol of the PDO  $\Psi_{\mathbf{x}}$  to be defined later and where we use bold  $\mathbf{x}$  subscript to emphasize the fact that  $\Psi_{\mathbf{x}}$  is defined on functions on  $\mathbb{R}^n$ . One can think of  $\Psi_{\mathbf{x}}$  as the directional version of  $\Psi_x$  with directional vector  $v = (1, \dots, 1)$ , this will be defined rigorously in Section 5.

Define the RL fractional derivative of index  $0 < \alpha < 1$  of  $f(\mathbf{x})$  to be

$$(2.7) \quad \mathfrak{D}_{\mathbf{x}}^{\alpha} f(\mathbf{x}) = \left( \sum_{i=1}^n \frac{\partial}{\partial x_i} \right) \int_0^{x_1} f(x_1 - r, x_2 - r, \dots, x_n - r) \frac{r^{-\alpha}}{\Gamma(1-\alpha)} dr.$$

Once again, Equation (2.7) can be thought of as a fractional directional derivative.

As opposed to the one dimensional case where under certain conditions the derivative w.r.t the time variable is defined on a function  $p(x; t)$ , in the finite dimensional case one can not avoid the fact that

$p(d\mathbf{x}; \mathbf{t})$  is  $f_{\mathbf{k}}(d\mathbf{x})$  valued on  $A_{\mathbf{k}}^n$ . In order to describe the dynamics of  $p(d\mathbf{x}; \mathbf{t})$  on  $A_{\mathbf{k}}^n$  one should extend this notion to the functions  $f_{\mathbf{k}}(\mathbf{t})$ , where we now use the letter  $\mathbf{t}$  in order to emphasize the context of this operator. Since on  $A_{\mathbf{k}}^n$  the dynamics on  $\mathbf{t}_{\mathbf{k}^c}$  are degenerate it is reasonable to apply  $\mathbb{D}_{\mathbf{x}}^{\alpha}$  on  $\mathbf{t}_{\mathbf{k}}$ . More precisely, if  $f_{\mathbf{k}}(d\mathbf{t}) = f(\mathbf{t}_{\mathbf{k}}) \delta_{\mathbf{t}_{\mathbf{k}^c}-1}(d\mathbf{t}_{\mathbf{k}^c}) d\mathbf{t}_{\mathbf{k}}$  (here  $f(\mathbf{t}_{\mathbf{k}})$  need not be a.c) then we define

$$\mathfrak{D}_{\mathbf{t}}^{\alpha} f_{\mathbf{k}}(d\mathbf{t}) := \mathbb{D}_{\mathbf{t}} \left[ \int_0^{x_1} f(x_{k_1} - r, \dots, x_{k_l} - r) \frac{r^{-\alpha}}{\Gamma(1-\alpha)} dr \delta_{\mathbf{t}_{\mathbf{k}^c}-1}(d\mathbf{t}_{\mathbf{k}^c}) \right].$$

The analogue of Lemma 2 is the following.

**Lemma 3.** *The LT of  $\mathfrak{D}_{\mathbf{t}}^{\alpha} f_{\mathbf{k}}(d\mathbf{t})$  is  $(\sum_{i=1}^n s_i)^{\alpha} \hat{f}_{\mathbf{k}}(\mathbf{s})$ .*

*Proof.* As before, we start with  $f_{\mathbf{k}}(d\mathbf{t})$  where  $l = n$  so that  $f_{\mathbf{k}}(d\mathbf{t}) = f(\mathbf{t})$ . A simple computation shows that

$$\mathcal{L} \left( \int_0^{t_1} f(t_1 - r, t_2 - r, \dots, t_n - r) \frac{r^{-\alpha}}{\Gamma(1-\alpha)} dr \right) (\mathbf{s}) = \left( \sum_{i=1}^n s_i \right)^{\alpha-1} \hat{f}(s_1, \dots, s_n).$$

Next, note that

$$\begin{aligned} \mathcal{L} \left( \int_0^{t_1} f(t_1 - r, t_2 - r, \dots, t_n - r) \frac{r^{-\alpha}}{\Gamma(1-\alpha)} dr \right) (t_1, s_2, \dots, s_n) \\ = \int_{r=0}^{t_1} e^{-(\sum_{i=2}^n s_i)r} \hat{f}(t_1 - r, s_2, \dots, s_n) dr \end{aligned}$$

so that  $\lim_{t_1 \rightarrow 0^+} \mathcal{L} \left( \int_0^{t_1} f(t_1 - r, t_2 - r, \dots, t_n - r) \frac{r^{-\alpha}}{\Gamma(1-\alpha)} dr \right) (t_1, s_2, \dots, s_n) = 0$ . It follows by Lemma 1 that

$$\widehat{\mathfrak{D}_{\mathbf{t}}^{\alpha} f_{\mathbf{k}}(d\mathbf{t})} = \left( \sum_{i=1}^n s_i \right)^{\alpha} \hat{f}(s_1, \dots, s_n).$$

The case where  $l < n$  is similar to Lemma 2. □

*Remark 1.* There is nothing exceptional about the operator  $\mathfrak{D}_{\mathbf{t}}^{\alpha}$ , in fact it is better to think of it as an archetype of PDOs corresponding to Laplace symbols of Lévy measures on  $\mathbb{R}_+$ . Indeed, if  $\phi(s) =$

$$\int_{\mathbb{R}_+} (e^{-sy} - 1) K_2(y) dy, \text{ then } \phi(s) \text{ is the symbol of the PDO } \Phi_t(f)(t) = \int_0^{\infty} (f(t-y) - f(t)) K_2(y) dy.$$

A simple calculation then shows that  $\phi(\sum_{i=1}^n s_i)$  is the symbol of  $\Phi_{\mathbf{t}}(f)(\mathbf{t}) = \int_0^{\infty} (f(t_1 - y, \dots, t_n - y) - f(\mathbf{t})) K_2(y) dy$ .

. The extension to the functions  $f_{\mathbf{k}}$  is obtained along similar lines to Lemma 3.

**2.4. The Semi-Markov Approach.** Since the process  $X_t = A_{E_t}$  is not Markovian, knowing its one dimensional distribution is not enough to construct its FDDs. To circumvent this problem Meerschaert and Straka ([10]) constructed the Markov process  $(X_t, R_t)$ , where  $R_t = D_{E_t} - t$  is the time left before the next regeneration of the process  $X_t$ . Let  $Q_t(x', r'; dx, dr)$  be the transition probability of the process  $(X_t, R_t)$  and  $0 < t_1 < t_2 < \dots < t_n$  for some  $n \in \mathbb{N}$ . Then

$$(2.8) \quad \begin{aligned} P(X_{t_1} \in dx_1, X_{t_2} \in dx_2, \dots, X_{t_n} \in dx_n) \\ = \int_{r_1=0}^{\infty} \int_{r_2=0}^{\infty} \dots \int_{r_n=0}^{\infty} Q_{t_1}(0, 0; dx_1, dr_1) \\ \times Q_{t_2-t_1}(x_1, r_1; dx_2, dr_2) \dots \times Q_{t_n-t_{n-1}}(x_{n-1}, r_{n-1}; dx_n, dr_n) \\ = Q_{t_1}(0, 0; dx_1, dr_1) \circ Q_{t_2-t_1}(x_1, r_1; dx_2, dr_2) \dots Q_{t_n-t_{n-1}}(x_{n-1}, r_{n-1}; dx_n, dr_n) \circ. \end{aligned}$$

Here,  $Q_t(x', r'; dx, dr) \circ f(x, r) = \int_{r=0}^{\infty} f(x, r) Q_t(x', r'; dx, dr)$  and  $Q_t(x', r'; dx, dr) \circ = \int_{r=0}^{\infty} Q_t(x', r'; dx, dr)$ .

In [10], the expression for  $Q_t$  is given for a large class of jump diffusions. Here, however, unless stated otherwise we consider processes of the form  $X_t = A_{E_t}$ , where  $A_t$  is a Lévy process and  $E_t$  is the inverse of a strictly increasing subordinator  $D_t$  that is independent of  $A_t$ . That is,

$$E_t = \inf \{s > 0 : D_s > t\}.$$

More precisely, the characteristic function of  $A_t$  and the Laplace transform of  $D_t$  are given respectively by

$$(2.9) \quad \begin{aligned} E(e^{ikA_t}) &= \exp \left[ t \left( ibk - \frac{1}{2}ak^2 + \int_{\mathbb{R}} (e^{iky} - 1 - iky1_{\{|y|<1\}}) K_1(dy) \right) \right] \\ E(e^{-sD_t}) &= \exp \left[ t \left( \int_{\mathbb{R}_+} (e^{-sy} - 1) K_2(dy) \right) \right]. \end{aligned}$$

Here,  $a \geq 0, b \in \mathbb{R}$ .  $K_1$  is a Lévy measure while  $K_2$  is a measure whose support is  $[0, \infty)$  and satisfies  $\int (y \wedge 1) K_2(dy) < \infty$ ,  $K_2(\{0\}) = 0$  and  $\int K_2(dy) = \infty$ . By (2.9) it can be easily verified that the infinitesimal generator  $\mathcal{A}$  of the process  $(A_t, D_t)$  is

$$(2.10) \quad \begin{aligned} \mathcal{A}(f)(x, t) &= b \frac{\partial}{\partial x} f(x, t) + \frac{a}{2} \frac{\partial^2}{\partial x^2} f(x, t) \\ &\quad + \int_{\mathbb{R}^2} \left( f(x + y, t + w) - f(x, t) - y \frac{\partial f(x, t)}{\partial x} 1_{\{|(y, w)| < 1\}} \right) K(dy, dw), \end{aligned}$$

where  $K$  is again a Lévy measure. In [10], the case where the coefficients  $b$  and  $a$  as well as the measure  $K$  may be dependent on  $(x, t)$  is considered. However, when they do not (this is referred to as the homogeneous case), the transition probability  $Q_t$  is given by ([10, Equation. 4.4])

$$(2.11) \quad \begin{aligned} Q_t(x', r'; dx, dr) &= 1_{\{0 < t < r'\}} \delta_0(dx - x') \delta_{r'-t}(dr) + 1_{\{0 \leq r' \leq t\}} Q_{t-r'}(x', 0; dx, dr) \\ Q_t(x', 0; dx, dr) &= \int_{y \in \mathbb{R}} \int_{w \in [0, t]} U^{x'}(dy, dw) K(dx - y, dr + t - w), \end{aligned}$$



where  $U^{x'}(dy, dw)$  is the occupation measure of  $(A_t, D_t)$ , i.e

$$\int f(y, w) U^{x'}(dy, dw) = \mathbb{E} \left( \int_0^\infty f(A_u + x', D_u) du \right).$$

When the processes  $A_t$  and  $D_t$  are independent, it can be easily verified that

$$(2.12) \quad U^{x'}(dy, dw) = \int_0^\infty z(dy - x', u) g(dw, u) du,$$

where  $z(dx, t) = P(A_t \in dx)$  and  $g(dx, t) = P(D_t \in dx)$ . Moreover, in the case of independence it was shown that ([4, Corollary 2.3])

$$K(dy, dw) = K_1(dy) \delta_0(dw) + \delta_0(dy) K_2(dw).$$

Hence, equations (2.11) translate into

$$(2.13) \quad \begin{aligned} Q_t(x', r'; dx, dr) &= 1_{\{0 < t < r'\}} \delta_0(dx - x') \delta_{r'-t}(dr) \\ &+ 1_{\{0 \leq r' \leq t\}} \int_{y \in \mathbb{R}} \int_{w \in [0, t-r']} \left( \int_0^\infty z(dy - x', u) g(dw, u) du \right) \\ &\times (\delta_0(dr + t - r' - w) K_1(dx - y) + \delta_0(dx - y) K_2(dr + t - r' - w)). \end{aligned}$$

However, since  $\int K_2(dy) = \infty$ , we see ([?, Theorem. 27.4]) that  $g(dw, t)$  has no atoms. Therefore, (2.13) reduces to

$$(2.14) \quad \begin{aligned} Q_t(x', r'; dx, dr) &= 1_{\{0 \leq t < r'\}} \delta_0(dx - x') \delta_{r'-t}(dr) \\ &+ 1_{\{0 \leq r' \leq t\}} \int_{w \in [0, t-r']} \left( \int_0^\infty z(dx - x', u) g(dw, u) du \right) \\ &\times K_2(dr + t - r' - w). \end{aligned}$$

### 3. FOKKER-PLANCK EQUATIONS

Throughout this section, we let  $A_t$  be a Lévy process such that  $E(e^{ikA_t}) = e^{t\psi(k)}$ , its probability density is given by  $z(dx, t) = P(A_t \in dx)$ .  $E_t$  is the inverse of a subordinator  $D_t$  such that  $E(e^{-sD_t}) = e^{t\phi(s)}$ , its probability density is  $h(dx, t) = P(E_t \in dx)$ . We denote by  $\Psi$  and  $\Phi$  the pseudo-differential operators of the symbols  $\psi(-k)$  and  $-\phi(s)$  respectively. We also denote the transition probability function of the Markov process  $(X_t, R_t)$  by  $Q_t$  and that of  $(E_t, R_t)$  by  $H_t$ . Next note that the occupation measure of  $(t, E_t)$  is just  $U^{x'}(dx, dw) = g(dw, x - x') dx$  (cf. [10, Eq. 5.1]), and similarly to (2.14) we have

$$(3.1) \quad \begin{aligned} H_t(x', r'; dx, dr) &= 1_{\{0 \leq t < r'\}} \delta_0(dx - x') \delta_{r'-t}(dr) \\ &+ 1_{\{0 \leq r' \leq t\}} \int_{w \in [0, t-r']} g(dw, x - x') dx \times K_2(dr + t - r' - w). \end{aligned}$$

The next theorem finds the FFPE of the FDD of  $E_t$ .

**Theorem 1.** Let  $h(dx_1, \dots, dx_n; t_1, \dots, t_n)$  be the FDD of  $E_t$  where  $t_1 < t_2 < \dots < t_n$ , i.e

$$h(dx_1, \dots, dx_n; t_1, \dots, t_n) = P(E_{t_1} \in dx_1, \dots, E_{t_n} \in dx_n).$$

Then

$$(3.2) \quad \Phi_{\mathbf{t}} h(d\mathbf{x}; \mathbf{t}) = -\mathbb{D}_{\mathbf{x}} h(d\mathbf{x}; \mathbf{t}).$$

*Proof.* Let us take LT with respect to the spatial variables and with respect to the time variables, this will be abbreviated by LLT. Before taking the LLT of  $h(d\mathbf{x}; \mathbf{t})$  we note that since  $H_t(x', r'; dx, dr)$  is translation invariant with respect to the spatial variable we have

$$(3.3) \quad \begin{aligned} & h(d\mathbf{x}; \mathbf{t}) \\ &= H_{t_1}(0, 0; dx_1, dr_1) \circ H_{t_2-t_1}(0, r_1; dx_2 - x_1, dr_2) \cdots H_{t_n-t_{n-1}}(0, r_{n-1}; dx_n - x_{n-1}, dr_n) \circ . \end{aligned}$$

Taking the LLT of (3.3), by a simple change of variables we see that (to avoid confusion we now use  $\lambda$  instead of  $k$ )

$$(3.4) \quad \begin{aligned} & \hat{h}(\lambda_1, \dots, \lambda_n; s_1, \dots, s_n) \\ &= \int_{t_1=0}^{\infty} \int_{x_1=0}^{\infty} e^{-(\sum_{i=1}^n s_i)t_1 - (\sum_{i=1}^n \lambda_i)x_1} H_{t_1}(0, 0; dx_1, dr_1) \circ dt_1 \\ & \quad \hat{H}_{\sum_{i=2}^n s_i} \left( 0, r_1; \sum_{i=2}^n \lambda_i, dr_2 \right) \circ \cdots \hat{H}_{s_n+s_{n-1}}(0, r_{n-2}; \lambda_n + \lambda_{n-1}, dr_{n-1}) \circ \hat{H}_{s_n}(0, r_{n-1}; \lambda_n, dr_n) \circ . \end{aligned}$$

Now, let us look at

$$\begin{aligned}
& \int_{t_1=0}^{\infty} \int_{x_1=0}^{\infty} e^{-(\sum_{i=1}^n s_i)t_1 - (\sum_{i=1}^n \lambda_i)x_1} H_{t_1}(0, 0; dx_1, dr_1) dt_1 \\
&= \int_{t_1=0}^{\infty} \int_{x_1=0}^{\infty} e^{-(\sum_{i=1}^n s_i)t_1 - (\sum_{i=1}^n \lambda_i)x_1} \int_{w \in [0, t_1]} g(w, x_1) dx_1 \\
&\quad \times K_2(dr_1 + t_1 - w) dw \\
&= \int_{x_1=0}^{\infty} e^{-(\sum_{i=1}^n \lambda_i)x_1} \int_{w \in [0, \infty]} g(w, x_1) dx_1 \\
&\quad \times \int_{t_1=w}^{\infty} e^{-(\sum_{i=1}^n s_i)t_1} K_2(dr_1 + t_1 - w) dw \\
&= \int_{x_1=0}^{\infty} e^{-(\sum_{i=1}^n \lambda_i)x_1} \int_{w \in [0, \infty]} g(w, x_1) dx_1 e^{-(\sum_{i=1}^n s_i)w} dw \\
&\quad \times \int_{t_1=0}^{\infty} e^{-(\sum_{i=1}^n s_i)t_1} K_2(dr_1 + t_1) dt_1 \\
(3.5) \quad &= \frac{1}{\sum_{i=1}^n \lambda_i - \phi(\sum_{i=1}^n s_i)} \int_{t_1=0}^{\infty} e^{-(\sum_{i=1}^n s_i)t_1} K_2(dr_1 + t_1) dt_1.
\end{aligned}$$

Next note that,

$$\begin{aligned}
(3.6) \quad & \lim_{x_1 \rightarrow 0^+} \hat{h}(x_1, \lambda_2, \dots, \lambda_n; s_1, \dots, s_n) \\
&= \lim_{x_1 \rightarrow 0^+} \int_{t_1=0}^{\infty} e^{-(\sum_{i=1}^n s_i)t_1 - (\sum_{i=2}^n \lambda_i)x_1} \int_{w \in [0, t_1]} g(dw, x_1) \times \int_{r_1=0}^{\infty} K_2(dr_1 + t_1 - w) \\
&\quad \times \hat{H}_{\sum_{i=2}^n s_i} \left( 0, r_1; \sum_{i=2}^n \lambda_i, dr_2 \right) \circ \dots \circ \hat{H}_{s_n + s_{n-1}}(0, r_{n-2}; \lambda_n + \lambda_{n-1}, dr_{n-1}) \circ \hat{H}_{s_n}(0, r_{n-1}; \lambda_n, dr_n) \circ . \\
&= \int_{t_1=0}^{\infty} e^{-(\sum_{i=1}^n s_i)t_1} \int_{r_1=0}^{\infty} K_2(dr_1 + t_1) \\
&\quad \times \hat{H}_{\sum_{i=2}^n s_i} \left( 0, r_1; \sum_{i=2}^n \lambda_i, dr_2 \right) \dots \hat{H}_{s_n + s_{n-1}}(0, r_{n-2}; \lambda_n + \lambda_{n-1}, dr_{n-1}) \circ \hat{H}_{s_n}(0, r_{n-1}; \lambda_n, dr_n) \circ .
\end{aligned}$$

Indeed, by the continuity of the measure  $K_2$  and [12, Lemma 27.1] follows the continuity of the following function

$$w \mapsto \int_{r_1=0}^{\infty} K_2(dr_1 + t_1 - w) \times \hat{H}_{\sum_{i=2}^n s_i} \left( 0, r_1; \sum_{i=2}^n \lambda_i, dr_2 \right) \circ \cdots \hat{H}_{s_n} (0, r_{n-1}; \lambda_n, dr_n) \circ,$$

since  $g(dw, x_1) dx_1$  converges weakly to  $\delta_0(dw)$  as  $x_1 \rightarrow 0^+$  (3.6) follows. Finally, plugging (3.5) in (3.4), using (3.6) and rearranging terms we arrive at

$$(3.7) \quad -\phi \left( \sum_{i=1}^n s_i \right) \hat{h}(\lambda_1, \dots, \lambda_n; s_1, \dots, s_n) = - \left( \sum_{i=1}^n \lambda_i \right) \hat{h}(\lambda_1, \dots, \lambda_n; s_1, \dots, s_n) + \hat{h}(0^+, \lambda_2, \dots, \lambda_n; s_1, \dots, s_n).$$

Taking the inverse LLT of (3.7) and using Lemma 2 we obtain (3.2).  $\square$

Theorem 1 paves the way for the finite dimensional FFPEs of the process  $X_t$ . We denote the FDD of  $A_t$  by  $z(dx_1, \dots, dx_n; t_1, \dots, t_n) = P(A_{t_1} \in dx_1, \dots, A_{t_n} \in dx_n)$ .

**Corollary 1.** *Let  $p(dx_1, \dots, dx_n; t_1, \dots, t_n) = P(X_{t_1} \in dx_1, \dots, X_{t_n} \in dx_n)$  where  $t_1 < t_2 < \dots < t_n$ . Then*

$$(3.8) \quad \begin{aligned} \Phi_t p(dx_1, \dots, dx_n; t_1, \dots, t_n) &= \Psi_{\mathbf{x}} p(dx_1, \dots, dx_n; t_1, \dots, t_n) \\ &+ \int_{u_2=0}^{\infty} \int_{u_3=u_2}^{\infty} \cdots \int_{u_n=u_{n-1}}^{\infty} \delta_0(dx_1) z(dx_2, \dots, dx_n; u_2, \dots, u_n) h(0^+, du_2, \dots, du_n; t_1, \dots, t_n) \end{aligned}$$

*Proof.* By the independence of  $A_t$  and  $D_t$

$$(3.9) \quad \begin{aligned} p(dx_1, \dots, dx_n; t_1, \dots, t_n) &= \int_{u_1=0}^{\infty} \int_{u_2=u_1}^{\infty} \cdots \int_{u_n=u_{n-1}}^{\infty} z(dx_1, \dots, dx_n; u_1, \dots, u_n) h(du_1, \dots, du_n; t_1, \dots, t_n) \\ &= \int_{u_1=0}^{\infty} \cdots \int_{u_n=u_{n-1}}^{\infty} z(dx_1, u_1) z(dx_2 - x_1, dx_3 - x_1, \dots, dx_n - x_1; u_2 - u_1, u_3 - u_1, \dots, u_n - u_1) \\ (3.10) \quad &\times H_{t_1}(0, 0; du_1, dr_1) \circ H_{t_2-t_1}(0, r_1; du_2 - u_1, dr_2) \circ \cdots H_{t_n-t_{n-1}}(0, r_{n-1}; du_n - u_{n-1}, dr_n) \circ. \end{aligned}$$

Taking the FLT of  $p(dx_1, \dots, dx_n; t_1, \dots, t_n)$  and using the change of variables  $u_i' = u_i - u_1$  for  $2 \leq i \leq n$  we obtain

$$\begin{aligned}
& \bar{p}(k_1, \dots, k_n; s_1, \dots, s_n) \\
&= \int_{u_1=0}^{\infty} \int_{t_1=0}^{\infty} \int_{x_1 \in \mathbb{R}} e^{-(\sum_{i=1}^n s_i)t_1 - (i \sum_{i=1}^n k_i)x_1} z(dx_1, u_1) H_{t_1}(0, 0; du_1, dr_1) \circ dt_1 \\
(3.11) \quad & \times \int_{u_2=0}^{\infty} \cdots \int_{u_n=u_{n-1}}^{\infty} \tilde{z}(k_2, \dots, k_n; u_2, \dots, u_n) \hat{H}_{\sum_{i=2}^n s_i}(0, r_1; du_2, dr_2) \circ \cdots \hat{H}_{s_n}(0, r_{n-1}; du_n - u_{n-1}, dr_n) \circ
\end{aligned}$$

Let us look at

$$\begin{aligned}
& \int_{u_1=0}^{\infty} \int_{t_1=0}^{\infty} \int_{x_1 \in \mathbb{R}} e^{-(\sum_{i=1}^n s_i)t_1 - (i \sum_{i=1}^n k_i)x_1} z(dx_1, u_1) H_{t_1}(0, 0; du_1, dr_1) dt_1 \\
&= \int_{u_1=0}^{\infty} \int_{t_1=0}^{\infty} \int_{x_1 \in \mathbb{R}} e^{-(\sum_{i=1}^n s_i)t_1 - (i \sum_{i=1}^n k_i)x_1} z(dx_1, u_1) \\
&\times \int_{w \in [0, t_1]} g(w, u_1) du_1 K_2(dr_1 + t_1 - w) dt_1 \\
&= \int_{u_1=0}^{\infty} \int_{w=0}^{\infty} \int_{x_1 \in \mathbb{R}} e^{-i(\sum_{i=1}^n k_i)x_1 - (\sum_{i=1}^n s_i)w} z(dx_1, u_1) g(w, u_1) du_1 \\
&\times \int_{t_1=0}^{\infty} e^{-(\sum_{i=1}^n s_i)t_1} K_2(dr_1 + t_1) dt_1 \\
(3.12) \quad &= \int_{u_1=0}^{\infty} \int_{w=0}^{\infty} \int_{x_1 \in \mathbb{R}} e^{u_1(\psi(-\sum_{i=1}^n k_i) + \phi(\sum_{i=1}^n s_i))} du_1 \times \int_{t_1=0}^{\infty} e^{-(\sum_{i=1}^n s_i)t_1} K_2(dr_1 + t_1) dt_1
\end{aligned}$$

$$(3.13) \quad = \frac{1}{-\psi(-\sum_{i=1}^n k_i) - \phi(\sum_{i=1}^n s_i)} \int_{t_1=0}^{\infty} e^{-(\sum_{i=1}^n s_i)t_1} K_2(dr_1 + t_1) dt_1.$$

Plugging (3.13) in (3.11) and using (3.6) we have

$$\begin{aligned}
\bar{p}(k_1, \dots, k_n; s_1, \dots, s_n) &= \frac{1}{-\psi(-\sum_{i=1}^n k_i) - \phi(\sum_{i=1}^n s_i)} \\
&\times \int_{u_2=0}^{\infty} \cdots \int_{u_n=u_{n-1}}^{\infty} \tilde{z}(k_2, \dots, k_n; u_2, \dots, u_n) \hat{h}(0^+, du_2, \dots, du_n; s_1, \dots, s_n).
\end{aligned}$$

Rearranging and taking the inverse FLT we arrive at (3.8).  $\square$

Working along similar lines to the proof of Theorem 1 one can also obtain the finite dimensional FFPEs of the process  $X_t = A_{E_t}$  where  $E_t$  is the inverse of a strictly increasing subordinator  $D_t$  and  $(A_t, D_t)$  is a Lévy process, i.e. the processes  $A_t$  and  $D_t$  are not necessarily independent. More precisely, suppose  $E(e^{ikA_t - sD_t}) = e^{t\xi(k, s)}$  and that  $\xi(k, s) = ibk - \frac{1}{2}ak^2 + \int_{\mathbb{R}} (e^{iky - sw} - 1 - iky1_{\{|(y, w)| < 1\}}) K(dy, dw)$  and that  $\Xi$  is the operator whose symbol is  $-\xi(-k, s)$ .

**Corollary 2.** *Let  $(A_t, D_t)$  be a Lévy process s.t  $E(e^{ikA_t - sD_t}) = e^{t\xi(k, s)}$ . Let  $E_t$  be the inverse of the strictly increasing subordinator  $D_t$  and let  $p(dx_1, \dots, dx_n; t_1, \dots, t_n) = P(X_{t_1} \in dx_1, \dots, X_{t_n} \in dx_n)$ . Then*

(3.14)

$$\begin{aligned} \Xi_{\mathbf{x}, \mathbf{t}} p(dx_1, \dots, dx_n; t_1, \dots, t_n) &= \int_{r_1=0}^{\infty} K(dx_1, dr_1 + t_1) \\ &\times Q_{t_2 - t_1}(x_1, r_1; dx_2, dr_2) \circ \dots \circ Q_{t_n - t_{n-1}}(x_{n-1}, r_{n-1}; dx_n, dr_n) \circ . \end{aligned}$$

*Proof.* Using (2.11) we see that  $Q_t$  is again translation invariant with respect to the spatial variable. Note that here

$$U^{x'}(dy, dw) = \int_0^{\infty} v(dy - x', dw; u) du,$$

where  $v(dy, dw; u) = P(A_u \in dy, D_u \in dw)$ . Using the same ideas as in the proof of Theorem 1 we obtain

(3.15)

$$\begin{aligned} \bar{p}(k_1, \dots, k_n; s_1, \dots, s_n) &= \int_{t_1=0}^{\infty} \int_{x_1 \in \mathbb{R}} e^{-(\sum_{i=1}^n s_i)t_1 - (\sum_{i=1}^n k_i)x_1} \int_{u=0}^{\infty} v(dy, dw; u) du \int_{r_1=0}^{\infty} \int_{y \in \mathbb{R}} \int_{w=0}^{t_1} K(dx_1 - y, dr_1 + t_1 - w) \\ &\times \bar{Q}_{\sum_{i=2}^n s_i} \left( 0, dr_1; \sum_{i=2}^n k_i, dr_2 \right) \circ \dots \circ \bar{Q}_{s_n} (0, r_{n-1}; k_n, dr_n) \circ \\ &= \int_{y \in \mathbb{R}} \int_{w=0}^{\infty} e^{-(\sum_{i=1}^n s_i)w - i(\sum_{i=1}^n k_i)y} \int_{u=0}^{\infty} v(dy, dw; u) du \int_{r_1=0}^{\infty} \int_{t_1=0}^{\infty} \int_{x_1 \in \mathbb{R}} e^{-(\sum_{i=1}^n s_i)t_1 - (\sum_{i=1}^n k_i)x_1} \\ &\times K(dx_1, dr_1 + t_1) \bar{Q}_{\sum_{i=2}^n s_i} \left( 0, dr_1; \sum_{i=2}^n k_i, dr_2 \right) \circ \dots \circ \bar{Q}_{s_n} (0, r_{n-1}; k_n, dr_n) \circ \\ &= \frac{1}{-\xi(-\sum_{i=1}^n k_i, \sum_{i=1}^n s_i)} \int_{r_1=0}^{\infty} \int_{t_1=0}^{\infty} \int_{x_1 \in \mathbb{R}} e^{-(\sum_{i=1}^n s_i)t_1 - (\sum_{i=1}^n k_i)x_1} K(dx_1, dr_1 + t_1) \\ &\times \bar{Q}_{\sum_{i=2}^n s_i} \left( 0, dr_1; \sum_{i=2}^n k_i, dr_2 \right) \circ \dots \circ \bar{Q}_{s_n} (0, r_{n-1}; k_n, dr_n) \circ . \end{aligned}$$

Rearrange and invert to obtain (3.14). □

*Remark 3.* As was mentioned above, it is usually impossible to define CTRWL by their one-dimensional FFPE. However, since Equation (3.14) and Equation (3.15) are equivalent and c̄iζœdliζœg processes are characterized (up to their law) by their FDDs, we see that one can define the process  $A_{E_t}$  by specifying all its  $n$  dimensional FFPE.

**Proposition 1.** *Let  $A_t$  and  $D_t$  as in Corollary 2. Then*

*Proof.* Let  $A'_t = (A_t, t)$  and note that  $A'_{E_t} = (X_t, E_t)$ . Using [10, Equation 4.4] (which is Equation (2.11) for outer process in  $\mathbb{R}^d$ ) for every  $x_1 \in \mathbb{R}$  we have

It is not hard to see that here

Plugging (3.18) in (3.17) we have

Integrating w.r.t  $x_2$  on  $[0, q]$  for some  $q > 0$  we have

$$\begin{aligned}
 P(X_t \in (-\infty, x_1], E_t \leq q) &= \int_{y_1 \in \mathbb{R}} \int_{w=0}^t \int_{u=0}^\infty v(dy_1, dw; u) du K((-\infty, x_1 - y_1], [t - w, \infty)) \mathbf{1}_{\{u \leq q\}} \\
 (3.20) \qquad \qquad \qquad &= \int_{y_1 \in \mathbb{R}} \int_{w=0}^t \int_{u=0}^q v(dy_1, dw; u) du K((-\infty, x_1 - y_1], [t - w, \infty)) .
 \end{aligned}$$

Taking derivative w.r.t  $q$  we have

$$\frac{\partial}{\partial q} P(X_t \in (-\infty, x_1], E_t \leq q) = \int_{y_1 \in \mathbb{R}} \int_{w=0}^t v(dy_1, dw; q) du K((-\infty, x_1 - y_1], [t - w, \infty)).$$

The measure  $K(dx_1, dw)$  is continuous because  $K(\mathbb{R}, dw) = K_2(dw)$  is continuous. Letting  $q \rightarrow 0$  we see that  $v(dy_1, dw; q) \xrightarrow{w} \delta_{(0,0)}(dy_1, dw)$  and hence  $\frac{\partial}{\partial q} P(X_t \in (-\infty, x_1], E_t \leq q) \rightarrow K((-\infty, x_1], [t, \infty))$  as  $q \rightarrow 0$  which is equivalent to (3.16).  $\square$

Proposition 1 sheds light on the “remainder” term that appears in Corollary 1 and Corollary 2. It appears that using a so-called multidimensional RL PDO in the finite dimensional FFPE one is left with a term that accounts for the portion of particles that have not been mobilized since  $t = 0$ . More philosophically, if we think of the value of  $E_t$  as the number of mobilizations of the process by time  $t$ , then  $\frac{\partial}{\partial u} P(E_t \leq u)$  is the ratio between the portion of particles  $\Delta m$  that experienced between  $u$  and  $u + \Delta u$  mobilizations up to time  $t$ . Evaluating  $\frac{\partial}{\partial u} P(E_t \leq u)$  at  $u = 0$  is then the ratio between the portion of particles  $\Delta m$  that experienced an infinitesimal number of mobilizations  $\Delta u$  by time  $t$  and  $\Delta u$ . If  $\frac{\partial}{\partial u} P(E_t \leq u)|_{u=0}$  is big then the diffusion becomes very dynamic at time  $t$  as many particles get loose and “take part” in the diffusion. Considering now  $\frac{\partial}{\partial u} P(X_t \in dx, E_t \leq u)|_{u=0}$  we see that since  $X_t$  is the limit of the Overshooting CTRW, where a jump precedes a waiting time the position of the particle that has been “stuck” until time  $t$  depends on that first jump in space. In that context is worthwhile to compare this to [6, Equation 4.2], the dynamics of the coupled CTRWL where the jump in space succeeds that in time. The “remainder” term therefore accounts for the Finite dimensional dynamics that only “kick in” at time  $t$ .

#### 4. EXAMPLES

Theorem 1 as well as Corollary 1 and Corollary 2 should be compared with their one-dimensional counterparts to gain a better understanding of the dynamics of the processes whose distributions govern the FFPE. We start with a specific case of the one dimensional analogue of Theorem 1.

**Example 1.** Let  $D_t$  be a standard stable subordinator of index  $0 < \alpha < 1$ , i.e.  $E(e^{-sD_t}) = e^{t(-s^\alpha)}$ . Its inverse  $E_t$  has a distribution  $h(x, t)$  which satisfies ([9, Equation 5.5])

$$\mathfrak{D}_t^\alpha h(x, t) = -\mathbb{D}_x h(x, t),$$

on  $x, t > 0$ . Since here  $\phi(s) = -s^\alpha$ , we see that  $\Phi_t = \mathfrak{D}_t^\alpha$ .

Next we look at the one dimensional analogue of Corollary 1.

**Example 2.** Again we let  $D_t$  be a standard stable subordinator of index  $0 < \alpha < 1$ , and  $A_t$  be a Lévy process s.t.  $E(e^{ikA_t}) = e^{t\psi(k)}$ . Then the distribution  $p(dx, t)$  of  $A_{E_t}$  satisfies ([9, Equation 5.6])

$$(4.1) \quad \mathfrak{D}_t^\alpha p(dx, t) = \Psi_x p(dx, t) + \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \delta_0(dx).$$

To see why (3.8) can be thought of as a generalization of (4.1) note that  $h(0^+, t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$  ([9, Equation 4.3]) and rewrite (4.1) as

$$\mathfrak{D}_t^\alpha p(x, t) = \Psi_x p(x, t) + \delta_0(dx) h(0^+, t).$$

Our last example concerns the one dimensional analogue of Corollary 2.



**Example 3.** Let  $(A_t, D_t)$  be a Lévy process as in Corollary 2. Then its one dimensional distribution  $p(dx, t)$  satisfies

$$(4.2) \quad \Xi_{x,t} p(dx, t) = \int_{r=0}^{\infty} K(dx, dr + t).$$

This was shown in [6, Theorem 4.1].

*Remark 4.* In [2, Equation 5.9], using different methods, Baule and Friedrich essentially obtained Equation (3.2) for the case where  $D_t$  is a standard stable subordinator. In [3, Equation 14] Baule and Friedrich obtained Equation (3.8)(uncoupled case) for the two dimensional case where  $D_t$  is a standard stable subordinator.

## 5. DIRECTIONAL PSEUDO-DIFFERENTIAL OPERATORS

In this section we wish to give a meaning to the PDO  $\Psi_{\mathbf{x}}, \Phi_{\mathbf{t}}$  and  $\Xi_{\mathbf{x}, \mathbf{t}}$  discussed earlier. We shall see that they are directional versions of their one-dimensional counterparts  $\Psi_x, \Phi_t$  and  $\Xi_{x,t}$ . We shall focus on  $\Xi_{\mathbf{x}, \mathbf{t}}$  as it is a generalization of  $\Psi_{\mathbf{x}}, \Phi_{\mathbf{t}}$ . To illustrate the kind of results we are looking for, let us look at the next simple example. Assume we have the following equation in  $\mathbb{R}^2$

$$(5.1) \quad \left( \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right) f(x_1, x_2) = h(x_1, x_2).$$

By using the change of variables  $(x_1, x_2)^T = \mathcal{T} (x'_1, x'_2)^T$  where  $\mathcal{T} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$  we can rewrite equation (5.1) as

$$\frac{\partial}{\partial x'_1} f(\mathcal{T} \mathbf{x}') = h(\mathcal{T} \mathbf{x}').$$

If we think of the change of variables  $\mathcal{T}$  as an operator on functions, i.e.  $\mathcal{T}f := f(\mathcal{T}\mathbf{x})$  we see that

$$\left( \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right) = \mathcal{T}^{-1} \frac{\partial}{\partial x'_1} \mathcal{T}.$$

Since the operator  $\mathbb{D}_{\mathbf{x}}$  is the classic one-dimensional derivative under the change of variables  $\mathcal{T}$  we say that it is a directional version of the classic derivative. We wish to show a similar result, i.e. that if  $\xi(-k, s)$  is a Lévy symbol, and therefore a symbol of a one-dimensional PDO  $\Xi_{x,t}$ , then  $\xi(-\sum_{i=1}^n k_i, \sum_{i=1}^n s_i)$  is the symbol of a PDO  $\Xi_{\mathbf{x}, \mathbf{t}}$  that is a directional version of  $\Xi_{x,t}$ .

In [1], the authors showed that if  $\xi(k, s)$  is a Lévy symbol then it is the symbol of a PDO on a Banach space. More precisely, Let  $X = L^1_{\omega}(\mathbb{R} \times \mathbb{R}_+)$  be the space of measurable functions s.t

$$\|f\|_{\omega} = \int_{\mathbb{R} \times \mathbb{R}_+} |f(x, t)| e^{-\omega t} dt dx < \infty \text{ where } \omega > 0 \text{ is fixed.}$$

With this norm, the space  $X$  is a Banach

space and the FLT is defined for each  $f \in X$  for  $k \in \mathbb{R}, s \in (\omega, \infty)$ . Let  $\xi(k, s)$  be a Lévy symbol, then it was shown that  $\xi(-k, s)$  is the symbol of the generator  $L$  of a Feller semigroup on  $X$ . Moreover, the domain of  $L$  is given by

$$D(L) = \{f \in X : \xi(-k, s) \overline{f}(k, s) = \overline{h}(k, s), \exists h \in X\}.$$

Let  $L^1_{\omega}(\mathbb{R}^n \times \mathbb{R}^n)$  denote the space of measurable functions that are defined on  $\mathbb{R}^n \times \mathbb{R}^n$  and

$$\int \int_{\mathbb{R}^n \mathbb{R}^n} |f(\mathbf{x}, \mathbf{t})| e^{-\langle \omega, \mathbf{t} \rangle} d\mathbf{x} d\mathbf{t} < \infty,$$

for some  $\omega \in \mathbb{R}^n_+$ . Let  $\mathcal{A}_n \subset L^1_{\omega}(\mathbb{R}^n \times \mathbb{R}^n)$  be the set of functions that vanish outside  $\mathbb{R}^n \times A^n$  where  $A^n = \{\mathbf{t} : 0 < t_1 \leq t_2 \leq \dots \leq t_n < \infty\}$ . Note that  $\mathcal{A}_n$  is itself a Banach space and that the FLT of  $f \in \mathcal{A}_n$  is defined for  $\mathbf{k} \in \mathbb{R}^n$ ,  $\mathbf{s} \in \hat{A}^n := \{\mathbf{s} : \omega_i < s_i\}$ . Let  $\mathcal{T}$  be the element of  $\text{GL}_n(n \times n$  invertible matrices) s.t its first column is  $(1, 1, \dots, 1)^T$  and its  $i$ 'th column  $c_i(j)$  is  $\delta_i(j)$  (1 if  $i = j$  and zero otherwise) for  $1 < i \leq n$ . Note that  $\det(\mathcal{T}) = 1$  so that  $\mathcal{T}$  is a bijection from the set  $B^n = \{(x_1, x_2, \dots, x_n) : x_1 > 0, 0 \leq x_2 \leq x_3 \leq \dots \leq x_n\}$  onto  $A^n$ . We introduce the change of variables  $\mathcal{T}\mathbf{x}' = \mathbf{x}$  and  $\mathcal{T}\mathbf{t}' = \mathbf{t}$  and see that for  $f \in \mathcal{A}_n$

$$\begin{aligned} (5.2) \quad \int \int_{\mathbb{R}^n \mathbb{R}^n_+} |f(\mathbf{x}, \mathbf{t})| e^{-\langle \omega, \mathbf{t} \rangle} d\mathbf{x} d\mathbf{t} &= \int \int_{\mathbb{R}^n \mathbb{R}^n_+} |f(\mathcal{T}\mathbf{x}', \mathcal{T}\mathbf{t}')| e^{-\langle \omega, \mathcal{T}\mathbf{t}' \rangle} \det(\mathcal{T})^2 d\mathbf{x}' d\mathbf{t}' \\ &= \int \int_{\mathbb{R}^n \mathbb{R}^n_+} |f(\mathcal{T}\mathbf{x}', \mathcal{T}\mathbf{t}')| e^{-\langle \mathcal{T}^* \omega, \mathbf{t}' \rangle} d\mathbf{x}' d\mathbf{t}', \end{aligned}$$

where  $\mathcal{T}^*$  is the adjoint of  $\mathcal{T}$ . Note that  $\mathcal{T}^*$  is the matrix whose first row is  $(1, 1, \dots, 1)$  and its  $i$ 'th row  $r_i(j)$  is  $\delta_i(j)$  for  $1 < i \leq n$ . Let  $\mathcal{B}_n \subset L^1_{\mathcal{T}^* \omega}(\mathbb{R}^n \times \mathbb{R}^n)$  be the subspace of functions that vanish outside  $\mathbb{R}^n \times B^n$  and note that it is a Banach space w.r.t the norm  $\|f\|_{\mathcal{T}^* \omega}$  and that its FLT is defined for  $(\mathbf{k}; \mathbf{s}) \in \mathbb{R}^n \times \hat{B}^n$  where  $\hat{B}^n := \mathcal{T}^* \hat{A}^n$ . We can now define the operator  $\mathcal{T} : \mathcal{A}_n \rightarrow \mathcal{B}_n$  by  $(\mathcal{T}f)(\mathbf{x}', \mathbf{t}') = f(\mathcal{T}\mathbf{x}', \mathcal{T}\mathbf{t}')$ . Note that by (5.2)  $\mathcal{T}$  is an isometric isomorphism from  $\mathcal{A}_n$  to  $\mathcal{B}_n$  and that if  $fg \in \mathcal{A}_n$  then  $\mathcal{T}(fg) = \mathcal{T}(f)\mathcal{T}(g) \in \mathcal{B}_n$ . We abuse notation and define the operator  $\Xi_{x,t} : \mathcal{B}^n \rightarrow \mathcal{B}^n$  by  $f(\mathbf{x}; \mathbf{t}) \mapsto \Xi_{x,t} f(\cdot, x_2, \dots, x_n; \cdot, t_2, \dots, t_n)$ . Finally, we define the operator  $\Xi_{\mathbf{x}, \mathbf{t}} : \mathcal{A}^n \rightarrow \mathcal{A}^n$  by  $\Xi_{\mathbf{x}, \mathbf{t}} = \mathcal{T}^{-1} \Xi_{x,t} \mathcal{T}$ .

**Proposition 2.** *Let  $\xi(-k, s)$  be a Lévy symbol of the  $n$  dimensional PDO  $\Xi_{x,t}$ , then  $\Xi_{\mathbf{x}, \mathbf{t}}$  is a PDO with symbol  $\xi(-\sum_{i=1}^n k_i, \sum_{i=1}^n s_i)$  and its domain is*

$$(5.3) \quad D(\Xi_{\mathbf{x}, \mathbf{t}}) = \left\{ f \in \mathcal{A}_n : \xi \left( -\sum_{i=1}^n k_i, \sum_{i=1}^n s_i \right) \bar{f}(\mathbf{k}, \mathbf{s}) = \bar{h}(\mathbf{k}, \mathbf{s}), \exists h \in \mathcal{A}_n \right\}.$$

*Proof.* . By (5.2) and Fubini's Theorem we have

$$(5.4) \quad \int \int_{\mathbb{R} \mathbb{R}_+} |\mathcal{T}f(x_1, x_2, \dots, x_n; t_1, \dots, t_n)| e^{-\langle \mathcal{T}^* \omega, (t_1, t_2, \dots, t_n) \rangle} dx'_1 dt'_1 < \infty,$$

and we see that  $\mathcal{T}f(\cdot, x_2, \dots, x_n; \cdot, t_2, \dots, t_n) \in L^1_{\sum_{i=1}^n \omega_i}(\mathbb{R} \times \mathbb{R})$  for almost every  $(x_2, \dots, x_n; t_2, \dots, t_n) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ . On the other hand, introducing  $\mathcal{T}^* \mathbf{k} = \mathbf{k}'$  and  $\mathcal{T}^* \mathbf{s} = \mathbf{s}'$  for  $(\mathbf{k}'; \mathbf{s}') \in \mathbb{R}^n \times \hat{B}^n$  and

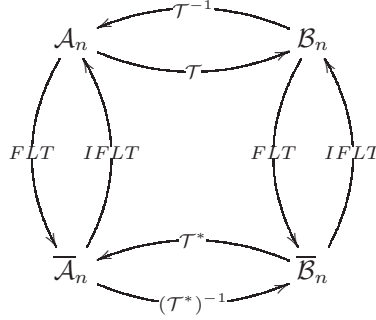
$f \in \mathcal{A}_n$  we have,

$$\begin{aligned}
\overline{f} \left( (\mathcal{T}^*)^{-1} \mathbf{k}', (\mathcal{T}^*)^{-1} \mathbf{s}' \right) &= \int \int_{\mathbb{R}^n \mathbb{R}_+^n} e^{-i \langle (\mathcal{T}^*)^{-1} \mathbf{k}', \mathbf{x} \rangle - \langle (\mathcal{T}^*)^{-1} \mathbf{s}', \mathbf{t} \rangle} f(\mathbf{x}, \mathbf{t}) d\mathbf{x} d\mathbf{t} \\
&= \int \int_{\mathbb{R}^n \mathbb{R}_+^n} e^{-i \langle \mathbf{k}', (\mathcal{T}^{-1}) \mathbf{x} \rangle - \langle \mathbf{s}', (\mathcal{T}^{-1}) \mathbf{t} \rangle} f(\mathbf{x}, \mathbf{t}) d\mathbf{x} d\mathbf{t} \\
&= \int \int_{\mathbb{R}^n \mathbb{R}_+^n} e^{-i \langle \mathbf{k}', \mathbf{x}' \rangle - \langle \mathbf{s}', \mathbf{t}' \rangle} f(\mathcal{T} \mathbf{x}', \mathcal{T} \mathbf{t}') \det(\mathcal{T}^{-1})^2 d\mathbf{x}' d\mathbf{t}' \\
&= \int \int_{\mathbb{R}^n \mathbb{R}_+^n} e^{-i \langle \mathbf{k}', \mathbf{x}' \rangle - \langle \mathbf{s}', \mathbf{t}' \rangle} \mathcal{T} f(\mathbf{x}', \mathbf{t}') d\mathbf{x}' d\mathbf{t}'.
\end{aligned}$$

It follows that for  $f \in \mathcal{A}_n$  on  $\mathbb{R}^n \times \hat{B}^n$  we have

$$(\mathcal{T}^*)^{-1} \overline{f} = \overline{\mathcal{T} f}.$$

To summarize, we have shown that the following diagram is commutative.



Here  $\overline{\mathcal{A}}_n$  and  $\overline{\mathcal{B}}_n$  denote the image of  $\mathcal{A}_n$  and  $\mathcal{B}_n$  respectively under the FLT map and IFLT is the inverse FLT. Note that  $\overline{\mathcal{A}}_n$  is defined on  $\mathbb{R}^n \times \hat{A}^n$  while  $\overline{\mathcal{B}}_n$  is defined on  $\mathbb{R}^n \times \hat{B}^n$ . Next, we note that the domain of  $\Xi_{x', t'}$  on  $\mathcal{B}_n$  is

$$D(\Xi_{x', t'}) = \{f : \xi(-k'_1, s'_1) f(\mathbf{k}', \mathbf{s}') = h(\mathbf{k}', \mathbf{s}'), \exists h \in \mathcal{B}_n\}.$$

Indeed, if  $f \in \mathcal{B}_n$  satisfies  $\xi(-k'_1, s'_1) f(\mathbf{k}', \mathbf{s}') = h(\mathbf{k}', \mathbf{s}')$  for some  $h \in \mathcal{B}_n$  then by (5.4) and the results in [1] we see that  $f \in D(\Xi_{x, t})$ . Next we show that  $\Xi_{\mathbf{x}, \mathbf{s}}$  is a PDO with symbol  $\psi(-\sum_{i=1}^n k_i, \sum_{i=1}^n s_i)$ . Suppose  $f \in D(\Xi_{\mathbf{x}, \mathbf{s}})$ , then

$$(5.5) \quad \Xi_{x', t'} \mathcal{T} f(\mathbf{x}', \mathbf{s}') = \mathcal{T} h(\mathbf{x}', \mathbf{s}'),$$

for some  $h \in \mathcal{A}_n$ . Applying FLT on both sides we obtain

$$(5.6) \quad \psi(-k'_1, s'_1) (\mathcal{T}^*)^{-1} \overline{f}(\mathbf{k}', \mathbf{s}') = (\mathcal{T}^*)^{-1} \overline{h}(\mathbf{k}', \mathbf{s}'),$$

multiplying both sides by  $\mathcal{T}^*$  we have

$$(5.7) \quad \psi\left(-\sum_{i=1}^n k_i, \sum_{i=1}^n s_i\right) \left[ \mathcal{T}^* (\mathcal{T}^*)^{-1} \overline{f}(\mathbf{k}, \mathbf{s}) \right] = \overline{h}(\mathbf{k}, \mathbf{s}).$$

Hence,  $\Xi_{\mathbf{x},\mathbf{s}}$  is a PDO with symbol  $\psi(-\sum_{i=1}^n k_i, \sum_{i=1}^n s_i)$ . It is left to show that the domain of  $\Xi_{\mathbf{x},\mathbf{s}}$  is as in (5.3). Since  $\mathcal{T}$  is a bijection it is clear that the following bijection holds  $\mathcal{T}D(\Xi_{\mathbf{x},\mathbf{t}}) = D(\Xi_{x,t})$  and the claim can be seen to be true through Equations (5.5), (5.6) and (5.7).  $\square$

In order to give a meaning to Equations (3.8) and (3.14) through Proposition 2 we define the function

$$f(\mathbf{x}; \mathbf{t}) = \int_{\mathbb{R}^n} g(\mathbf{x} - \mathbf{y}) p(d\mathbf{y}; \mathbf{t}),$$

where  $g$  is a smooth function with compact support in  $\mathbb{R}^n$  and  $p(d\mathbf{x}; \mathbf{t})$  is a parameterized distribution as in Subsection 2.1. It follows that  $f(\mathbf{x}; \mathbf{t})$  is smooth in  $\mathbb{R}^n$  for every  $\mathbf{t} \in \mathbb{R}_+^n$ . Multiply both sides of Equation (3.15) by  $\tilde{g}(\mathbf{k})$  and use the convolution-multiplication property of the FT to obtain

$$\Xi_{\mathbf{x},\mathbf{t}} f(\mathbf{x}; \mathbf{t}) = \int_{\mathbb{R}^n} g(\mathbf{x} - \mathbf{y}) p_0(d\mathbf{y}; \mathbf{t}),$$

where

$$\begin{aligned} p_0(d\mathbf{x}; \mathbf{t}) &= \int_{r_1=0}^{\infty} K(dx_1, dr_1 + t_1) \\ &\quad \times Q_{t_2-t_1}(x_1, r_1; dx_2, dr_2) \circ \cdots \circ Q_{t_n-t_{n-1}}(x_{n-1}, r_{n-1}; dx_n, dr_n) \circ. \end{aligned}$$

This interpretation of (3.8) and (3.14) is in the spirit of that in [11, Chapter 4] and was used in [5, p. 15].

## 6. CONCLUSIONS

In this paper we find the FDD's FFPEs of the process  $A_{E_t}$  where  $(A_t, D_t)$  is a Lévy process,  $D_t$  is a strictly increasing subordinator with no drift and  $E_t$  is the inverse of  $D_t$ . The general form of these FFPEs (Eq. 3.14) is a PDO in time and space variables applied to the distribution of the process on one side of the equation while on the other we have a term that accounts for the portion of particles that yet to be mobilized. Moreover, considering the difference between the RL derivative and that of Caputo's in the one dimensional case, and compared to the finite dimensional one, it seems that the RL derivative is more suitable in the context of CTRWL. We also showed that the PDO which appear in Corollary 2 are indeed bona fide PDO and in fact a directional version of their one dimensional counterparts.

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